Lebesgue Integration

The problems below are taken out of various textbooks on real variables, including "Real Analysis" by Elias M. Stein and Rami Shakarchi and "Real Analysis" by N. L. Carothers. Questions are also taken from real variables qualifying exams at CUNY Graduate Center. The problems are color-coded. The color green indicates that the problem came from a textbook and to the best of my knowledge was not featured on any qualifying exam. Yellow means that the problem was spotted in at least one qualifying exam. Red indicates that the problem or one just like it appeared in at least two qualifying exams.

1. Prove that if f is integrable on $\mathbf{R}^d$ and $\delta > 0$, then $f(\delta x)$ converges to $f(x)$ in the L^1-norm as $\delta \to 1$.

2. Suppose f is integrable on $(-\pi, \pi]$ and extended to $\mathbf{R}$ by making it periodic of period 2π. Show that
$$\int_{-\pi}^{\pi} f(x)dx = \int_I f(x)dx$$
where I is any interval in $\mathbf{R}$ of length 2π.

3. Suppose $f \in L^1[0, \infty)$ and for all $x \in [0, \infty)$, $|f(x)| < \pi/2$. Show that $\mathrm{Sin}(f) \in L^1[0, \infty)$, and
$$\int_0^{\infty} Sin^n(f(x))dx \to 0 \text{ as } x \to \infty.$$

4. Find a sequence f_n of nonnegative measurable functions such that $\lim_{n\to\infty} f_n = 0$, but $\lim_{n\to\infty} \int f_n = 1$. In fact, show that f_n can be chosen to converge *uniformly* to 0.

5. Let f be measurable with $f > 0$ a.e. If $\int_E f = 0$ for some measurable set E, show that $m(E) = 0$.

6. Integrability of f on $\mathbf{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.
   (a) There exists a positive continuous function f on $\mathbf{R}$ so that f is integrable on $\mathbf{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.
   (b) However, if we assume that f is uniformly continuous on $\mathbf{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

7. Let $g : [0, 1] \to [0, 1]$ be the Cantor function. Calculate $\int_0^1 g$.

8. Define f: [0, 1] → [0, ∞) by f(x) = 0 if x is rational and f(x) = 2^n if x is irrational with exactly n = 0, 1, 2, … leading zeros in its decimal expansion. Show that f is measurable, and find $\int_0^1 f$.

9. Let f: (0, ∞) → **R** be the function $f(x) = \frac{Sin(x)}{x}$. Prove that the (improper Riemann integral $\int_0^\infty f(x)dx$ exists, but that f is not Lebesgue integrable. (The common folklore is that the two integrals are the same when the Riemann integral exists; the example shows this can be false when the Riemann integral is "improper.")

10. Prove that $\int_0^\infty e^{-x}dx = \lim_{n\to\infty}\int_0^n (1-(x/n))^n dx = 1$.

11. Compute $\lim_{n\to\infty}\int_0^n (1-(x/n))^n e^{x/2}dx$, justifying your calculations.

12. Suppose f is integrable on [0, b], and

$$g(x) = \int_x^b \frac{f(y)}{y}dy \quad \text{for } 0 < x \le b.$$

Prove that g is integrable on [0, b] and

$$\int_0^b g(x)dx = \int_0^b f(y)dy .$$

13. Let $f(x) = x^{-1/2}$ for 0 < x < 1 and $f(x) = 0$ otherwise. Let $\{r_n\}$ be an enumeration of **Q**, and let $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show that:

(a) $g \in L^1(\mathbf{R})$ and, in particular, g is finite a.e.
(b) g is discontinuous at every point and is unbounded on every interval; it remains so even after modification on an arbitrary set of measure zero.
(c) g^2 is finite a.e., but g^2 is not integrable on any interval.

14. Suppose that E ⊂ [0, 2π] is measurable and that $\int_E x^n Cos(x)dx = 0$ for all n = 0, 1, 2, … Show that m(E) = 0

15. Discuss, giving reasons, for each of the two cases [a, b] = [0, 1], and [a, b] = [-1, 1], whether or not it is possible for two different continuous functions f and g on [a, b] to have the same even moments. I.e., $\int_a^b x^{2n} f(x)dx = \int_a^b x^{2n} g(x)dx$ (n = 0, 1, 2, …).

16. If f, $\{f_n\}$ are Lebesgue integrable, and if $\{f_n\}$ increases pointwise to f, does it follow that $\int f_n \to \int f$? Explain.

17. Let $\|x\|$ denote the distance of x from the nearest integer. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, and $0 < \alpha < 1$. Show that the series defining $f(x) = \sum_{n=1}^{\infty} a_n \|nx\|^{-\alpha}$ converges for almost all $x \in \mathbf{R}$.

18. If $f \in L^1[0, 1]$, show that $x^n f(x) \in L^1[0, 1]$ for n =1, 2, … and compute $\lim_{n\to\infty} \int_0^1 x^n f(x)dx$.

19. Compute $\sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{Sin(x)})^n Cos(x)dx$. Justify your calculations.

20. Let $\{f_n\}$, $\{g_n\}$, and g be integrable on $\mathbf{R}^d$, and suppose that $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \le g_n$ a.e., for all n, and that $\int g_n \to \int g$. Prove that $f \in L^1(\mathbf{R}^d)$ and that $\int f_n \to \int f$. (This is a variation of Lebesgue dominated convergence theorem.)

21. Let $\{g_n\}$ be a sequence of Lebesgue integrable functions on $\mathbf{R}^d$ and suppose that $g_n \to g$ a.e. and $\int g_n \to \int g$ where g is also Lebesgue integrable on $\mathbf{R}^d$. Answer the following:

(a) If A_n is an increasing sequence of measurable sets (i.e. $A_n \subset A_{n+1}$) with $\lim_{n\to\infty} A_n = \mathbf{R}^d$ and $g_n(x) \ge 0$ for almost every x, does it follow that

$$\int_{A_n} g_n \to \int g$$

(b) How about if we drop the assumption that g_n is nonnegative a.e.?

22. Let $\{f_n\}$ be a sequence of integrable functions and suppose that $|f_n| \le g$ a.e., for all n, for some integrable function g. Prove that

$$\int (\liminf_{n\to\infty} f_n) \le \liminf_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f_n \le \int (\limsup_{n\to\infty} f_n).$$

23. Let f be measurable and finite a.e. on [0, 1].

(a) If $\int_E f = 0$ for all measurable $E \subset [0, 1]$ with m(E) = ½, prove that $f = 0$ a.e. on [0, 1].

(b) If $f > 0$ a.e., show that $\inf\{\int_E f : m(E) \geq ½\} > 0$.

24. Show that $\lim_{n\to\infty} \int_0^1 f_n = 0$ where $f_n(x)$ is:

(a) $\dfrac{nx}{1+n^2x^2}$

(b) $\dfrac{n\sqrt{x}}{1+n^2x^2}$

(c) $\dfrac{nx\ln(x)}{1+n^2x^2}$

(d) $\dfrac{n^{3/2}x}{1+n^2x^2}$

25. Find:

(a) $\lim\limits_{n\to\infty} \displaystyle\int_0^\infty \frac{Sin(e^x)}{1+nx^2}dx$

(b) $\lim\limits_{n\to\infty} \displaystyle\int_0^1 \frac{nCos(x)}{1+n^2x^{3/2}}dx$

26. Fix $0 < a < b$, and define $f_n(x) = a^2e^{-nax} - b^2e^{-nbx}$. Show that $\sum_{n=1}^\infty \int_0^\infty |f_n| = \infty$ and $\int_0^\infty \left(\sum_{n=1}^\infty f_n\right) \neq \sum_{n=1}^\infty \int_0^\infty f_n$.

27. Compute the following limits, justifying your calculations:

(a) $\lim\limits_{n\to\infty} \displaystyle\int_0^\infty \frac{nSin(x/n)}{x(1+x^2)}dx$

(b) $\lim\limits_{n\to\infty} \displaystyle\int_0^1 \frac{1+nx^2}{(1+x^2)^n}dx$

(c) $\lim\limits_{n\to\infty} \displaystyle\int_0^\infty \frac{Sin(x/n)}{(1+x/n)^n}dx$

(d) $\lim\limits_{n\to\infty} \displaystyle\int_a^\infty \frac{n}{1+n^2x^2}dx$

28. Let E be a measurable subset of $\mathbf{R}^d$ with m(E) > 0 and let f: E → [0, ∞] be a measurable function. Prove that

$$\lim_{n\to\infty} \int_E n\ln(1+f/n) = \int_E f$$

$$\lim_{n\to\infty} \int_E n\ln(1+(f/n)^{1/2}) = \infty$$

29. Let $\alpha, \beta \in \mathbf{R}$, and define $f(x) = x^{\alpha} Sin(x^{\beta})$, $0 < x \le 1$. For what values of α and β is f: (i) Riemann integrable (in the sense that $\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} f(x)dx$ exists)? (ii) Lebesgue integrable?

30. For which $\alpha \in \mathbf{R}$ is $f(x) = \sum_{n=1}^{\infty} xn^{-\alpha} e^{-nx}$ continuous on $[0, \infty)$? in $L^1[0, \infty)$?

31. Let

$$f(x) = \sum_{n=1}^{\infty} (1/n) e^{-n(x-n)^2}$$

for $x \in \mathbf{R}$.

(a) Is f in $L^1(\mathbf{R})$?
(b) Is f continuous on $\mathbf{R}$?
(c) Is f differentiable on $\mathbf{R}$?

Solutions:

1. We can use the fact that continuous functions with compact support are dense in $L^1(\mathbf{R}^d)$ to establish the claim. In particular, let g be a continuous function on a compact support E that satisfies

$$\int | f(x) - g(x) | dx < \varepsilon .$$

Then for $\delta > 0$,

$$\int | f(\delta x) - g(\delta x) | dx = \delta^{-d} \int | f(x) - g(x) | dx < \delta^{-d} \varepsilon .$$

And we have

$$\int | f(\delta x) - f(x) | dx \le \int | f(\delta x) - g(\delta x) | dx + \int | g(\delta x) - g(x) | dx + \int | g(x) - f(x) | dx <$$
$$(\delta^{-d} + 1)\varepsilon + \int | g(\delta x) - g(x) | dx .$$

Thus, it suffices to prove

$$\lim_{\delta \to 1} \int | g(\delta x) - g(x) | dx = 0 .$$

To accomplish this, let Q be a closed cube big enough to contain E and 2E. This closed cube Q must then contain all sets of the form $\delta^{-1}E$ whenever $\delta > ½$ and therefore the functions $k_{\delta}(x) = | g(\delta x) - g(x) |$ are bounded above by 2M, where $M = \sup\{ | g(x) | : x \in E\}$, and supported on Q. By the bounded convergence theorem, we then have

$$\lim_{\delta \to 1} \int | g(\delta x) - g(x) | dx = \int \lim_{\delta \to 1} | g(\delta x) - g(x) | dx = 0 .$$

2. We may assume without loss of generality that I is of the form $(a - \pi, a + \pi]$, where

a = 2πk + r, k $\in$ **Z**, and $0 \le r < 2\pi$. It is easily derived from the properties of measurable sets that for any Lebesgue integrable function g on $\mathbf{R}^d$, one has

$$\int_{R^d} g(x)dx = \int_{R^d} g(x+h)dx$$

for $h \in \mathbf{R}^d$.

In particular, if g: (c, d) → **R**, then

$$\int_c^d g(x)dx = \int_R g(x)\chi_{(c,\ d)}(x)dx = \int_R g(x+h)\chi_{(c,\ d)}(x+h)dx =$$

$$\int_R g(x+h)\chi_{(c-h,\ d-h)}(x)dx = \int_{c-h}^{d-h} g(x+h)dx$$

Thus,

$$\int_I f(x)dx = \int_{a-\pi}^{a+\pi} f(x)dx = \int_{2\pi k+r-\pi}^{2\pi k+r+\pi} f(x)dx = \int_{r-\pi}^{r+\pi} f(x+2\pi k)dx \qquad (1)$$

Since f is periodic with period 2π, we get from (1)

$$\int_I f(x)dx = \int_{r-\pi}^{r+\pi} f(x)dx \qquad (2)$$

Finally, we break the integral in (2) to obtain

$$\int_I f(x)dx = \int_{r-\pi}^{r+\pi} f(x)dx = \int_{r-\pi}^{\pi} f(x)dx + \int_{\pi}^{r+\pi} f(x)dx =$$

$$\int_{r-\pi}^{\pi} f(x)dx + \int_{-\pi}^{r-\pi} f(x+2\pi)dx = \int_{-\pi}^{\pi} f(x)dx.$$

3. Recall that Sin(x) is a Lipschitz function that satisfies |Sin(x) – Sin(y)| ≤ |x – y|. Therefore |Sin(f(x))| = |Sin(f(x)) – Sin(0)| ≤ |f(x) – 0| = |f(x)|. By the assumption on f and monotonicity of the Lebesgue integral,

$$\int_0^\infty |Sin(f(x))|\,dx \le \int_0^\infty |f(x)|\,dx < \infty$$

This shows that Sin(f) $\in L^1[0, \infty)$.

The hypothesis that $|f(x)| < \pi/2$ implies |Sin(f(x))| < 1 for all x. Consequently,

$$\lim_{n\to\infty} Sin^n(f(x)) = 0 \text{ for all } x \in [0, \infty).$$

Moreover, since $|Sin^n(f(x))| \le |Sin(f(x))| \le |f(x)|$, we may apply Lebesgue dominated convergence theorem to conclude that

$$\int_0^\infty Sin^n(f(x))dx \to 0 \text{ as } x \to \infty.$$

4. Define $f_n(x) = \frac{1}{n}\chi_{(0,n)}(x)$, where $\chi_{(0,n)}$ is the indicator function of the interval (0, n).

Clearly $|f_n(x) - 0| \le \frac{1}{n}$ and therefore $f_n \to 0$ uniformly. However, $\int f_n = \frac{1}{n}n = 1$.

5. Since $f > 0$ a.e. the set E can we written as a union $E_0 \cup \bigcup_{n=1}^{\infty} E_n$ where $E_0 = \{x \in E: \ f(x) \le 0\}$ is of measure 0 and $E_n = \{x \in E: \ f(x) \ge 1/n\}$. Define $f_n = \frac{1}{n}\chi_{E\,n}$.
Then, by monotonicity of the integral,

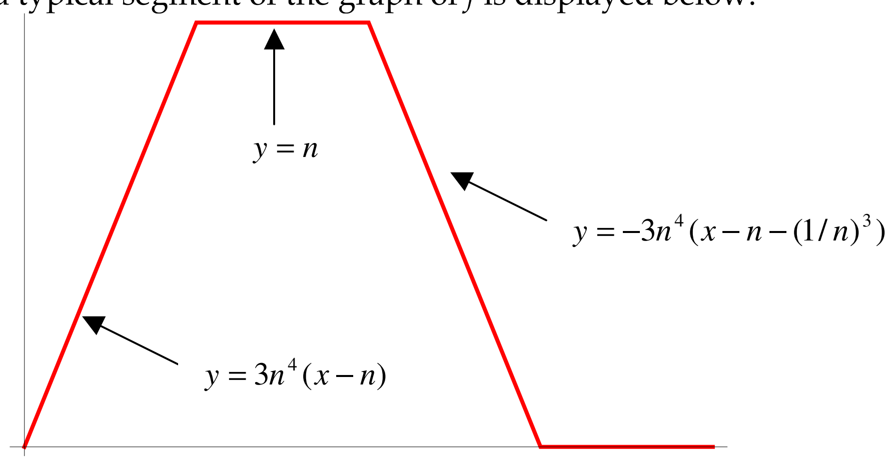
$$0 = \int_E f \ge \int_E f_n = \frac{1}{n} m(E_n).$$

Thus m(E_n) = 0 and therefore m(E) ≤ m(E_0) + $\sum_{n=1}^{\infty} m(E_n)$ = 0.

6. (a) Define for each n ∈ **N**

$$f(x) = \begin{cases} 3n^4(x-n) & x \in \left[n,\ n + \frac{1}{3}(1/n)^3\right] \\ n & x \in \left(n + \frac{1}{3}(1/n)^3,\ n + \frac{2}{3}(1/n)^3\right) \\ -3n^4(x - n - (1/n)^3) & x \in \left[n + \frac{2}{3}(1/n)^3,\ n + (1/n)^3\right] \\ 0 & otherwise \end{cases}$$

For convenience, a typical segment of the graph of *f* is displayed below:



Clearly, *f* is continuous on **R**. As n approaches infinity, the horizontal peak y = n becomes unbounded. Hence lim sup$_{x\to\infty} f$(x) = ∞.
Finally, to see that *f* is integrable, it suffices to note that *f* ≥ 0 and that the hill on the interval [n, n + 1] is bounded above by the function

$$n\chi_{[n,\ n+(1/n)^3]}.$$

Thus

$$\int f \le \sum_{n=1}^{\infty} \frac{1}{n^3} n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(b) Assume that f is uniformly continuous and Lebesgue integrable on $\mathbf{R}$. If we can prove that $\lim_{|x|\to\infty} f(x) = 0$ in the special case when $f \geq 0$, the general result will follow from the decomposition $f = f^{+} - f^{-}$, where $f^{+} = \max\{f, 0\}$ and $f^{-} = \max\{-f, 0\}$. To that end, notice that if $f \geq 0$ is uniformly continuous, but $\lim_{|x|\to\infty} f(x) \neq 0$, we may find an $\epsilon > 0$ and a sequence x_n such that $|x_n| > n$ and $f(x_n) \geq \epsilon$. Because f is uniformly continuous, we may also find a $\delta > 0$ such that $f(x) > \epsilon/2$ for all $x \in (x_n - \delta, x_n + \delta)$. We then have

$$\int f \geq \sum_{n=1}^{\infty} \frac{\varepsilon}{2} 2\delta = \infty .$$

Hence all integrable, uniformly continuous functions must vanish at infinity.

7. Recall that g is the extension of the function $c : \Delta \to [0, 1]$ defined on the Cantor set Δ. The function c is given by

$$c(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

where $x \in \Delta$ has the ternary base representation $0.(2a_1)(2a_2)...(2a_n)...$ (mod 3) and $a_i = 0$ or 1. The function g is consequently defined by $g(x) = \sup c(y)$, where the supremum is taken over all $y \in \Delta$ and $y \leq x$.
We can write

$$\int_0^1 g = \int_{\Delta} g + \int_{[0,1]-\Delta} g = \int_{[0,1]-\Delta} g$$

because Δ is a set of measure 0 and Lebesgue integrals vanish on such sets.
Notice that $[0, 1] - \Delta$ is the union of disjoint open intervals and that g is constant on each interval. Denote by L_n the collection of the 2^{n-1} disjoint open intervals in $[0, 1] - \Delta$ of length 3^{-n} that were deleted from $[0, 1]$ on the n^{th} step of the construction of the Cantor set. Let $x_{n,\ j}$; $1 \leq j \leq 2^{n-1}$ denote the left-endpoints of the intervals in L_n. Then

$$g = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} g(x_{n,\ j}) \chi_{I_{n,\ j}}$$

is the limit of an increasing sequence of step functions. Thus, by the monotone convergence theorem,

$$\int_{[0,1]-\Delta} g = \sum_{n=1}^{\infty} \frac{1}{3^n} \sum_{j=1}^{2^{n-1}} g(x_{n,\ j}) .$$

To compute

$$\sum_{j=1}^{2^{n-1}} g(x_{n,\ j})$$

observe that $x_{n,\ j}$ is of the form $0.(2a_1)(2a_2)...(2a_{n-1})$ (mod 3) where a_i = 0 or 1. Hence,

$$\sum_{j=1}^{2^{n-1}} g(x_{n,\ j}) = \sum_{(a_1,...,a_{n-1}):\ a_i=0,\ 1} \sum_{i=1}^{n-1} \frac{a_i}{2^i} + 2^{n-1}\frac{1}{2^n} =$$

$$\sum_{(a_1,...,a_{n-1}):\ a_i=0,\ 1} \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{1}{2}$$

where the sum is taken over all 2^{n-1} possible vectors $(a_1,..., a_{n-1})$ with entries 0 or 1. Let

$$S_n = \sum_{(a_1,...,a_{n-1}):\ a_i=0,\ 1} \sum_{i=1}^{n-1} \frac{a_i}{2^i}$$

Then

$$S_n = \sum_{(a_1,...,a_{n-2},0):\ a_i=0,\ 1} \sum_{i=1}^{n-2} \frac{a_i}{2^i} + \sum_{(a_1,...,a_{n-2},1):\ a_i=0,\ 1} \left(\sum_{i=1}^{n-2} \frac{a_i}{2^i} + \frac{1}{2^{n-1}} \right) =$$

$$S_{n-1} + S_{n-1} + 2^{n-2}\frac{1}{2^{n-1}} = 2S_{n-1} + \frac{1}{2}$$

Clearly, $S_1 = 0$ and the closed formula for S_n is therefore $S_n = \frac{1}{2}\sum_{k=0}^{n-2} 2^k = \frac{2^{n-1}-1}{2}$.

It follows that

$$\sum_{j=1}^{2^{n-1}} g(x_{n,\ j}) = S_n + \frac{1}{2} = 2^{n-2}$$

Thus,

$$\int_{[0,1]-\Delta} g = \sum_{n=1}^{\infty} \frac{2^{n-2}}{3^n} = \frac{1}{6}\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = \frac{1}{2}$$

8. For each n = 0, 1, 2, … define $E_n = [10^{-n-1}, 10^{-n}] \cap (\mathbf{R} - \mathbf{Q})$. Then it is clear that the function *f* is given by

$$f = \sum_{n=0}^{\infty} 2^n \chi_{E_n}$$

where χ_{E_n} is the indicator function of E_n. Since *f* is the limit of simple functions and therefore the pointwise limit of measurable functions, *f* is measurable. Applying the monotone convergence theorem, we obtain

$$\int_0^1 f = \sum_{n=0}^{\infty} 2^n \int_0^1 \chi_{E_n} = \sum_{n=0}^{\infty} 9\frac{2^n}{10^{n+1}} = \frac{9}{8}.$$

9. There are several ways to prove that $f(x) = \dfrac{Sin(x)}{x}$ is Riemann integrable over (0, ∞). One method is to calculate the integral's exact value by defining the function $g : [0, \infty) \to [-\infty, \infty]$ which is given by

$$g(t) = \int_0^{\infty} e^{-xt}\frac{Sin(x)}{x}dx = \int_0^{\infty}\int_t^{\infty} e^{-xy} Sin(x)dydx.$$

If we can express the right-hand-side of the formula in a familiar form of a known continuous function, the value $g(0) = \int_0^{\infty} f(x)dx$ will be the desired result. To that end, let t > 0 and set $k, h : [0, \infty) \times (t, \infty)$ to be the functions $k(x, \ y) = e^{-xy} Sin(x)$ and $h(x, \ y) = e^{-xy}x$ respectively. Then $|k| \le h$ and since *h* is nonnegative, we may apply Tonelli's theorem to establish that

$$\int_0^{\infty}\int_t^{\infty} h(x,y)dydx = \int_0^{\infty}\int_t^{\infty} e^{-xy}xdydx = \frac{1}{t} < \infty$$

and that therefore $h \in L^1([0, \infty) \times (t, \infty))$. This implies that $k \in L^1([0, \infty) \times (t, \infty))$ as well and by Fubini's theorem, we may change the order of integration to obtain

$$g(t) = \int_0^{\infty}\int_t^{\infty} e^{-xy} Sin(x)dydx = \int_t^{\infty}\int_0^{\infty} e^{-xy} Sin(x)dxdy = \int_t^{\infty}\frac{1}{1+y^2}dy = \frac{\pi}{2} - \tan^{-1}(t)$$

Hence *g* is continuous and $g(0) = \pi/2$.
Another slick method to prove the integral exists (borrowed from Carothers) is to write $\int_0^{\infty}\frac{Sin(x)}{x}dx$ as an alternating series:

$$\begin{aligned}\int_0^{\infty}\frac{Sin(x)}{x}dx &= \sum_{n=1}^{\infty}\int_{(n-1)\pi}^{n\pi}\frac{Sin(x)}{x}dx \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\int_{(n-1)\pi}^{n\pi}\frac{|Sin(x)|}{x}dx \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^{\pi}\frac{|Sin(x)|}{x+(n-1)\pi}dx.\end{aligned}$$

To show that the series converges, we only have to show that the terms tend monotonically to zero. But $|\mathrm{Sin}(x)|/(x + (n - 1)\pi)$ clearly decreases as n increases (for x fixed), and

$$\int_0^{\pi} \frac{|\,Sin(x)\,|}{x+(n-1)\pi}\,dx \le \frac{1}{n-1} \to 0.$$

To see that the Lebesgue integral does not exist, observe that the assertion "f is Lebesgue integrable on $(0, \infty)$" is equivalent to the assertion "f is in $L^1(0, \infty)$". That is, Lebesgue integrability implies the integrability of the function $|f| = f^+ + f^-$ and hence the integrability of $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. It therefore suffices to show that f^+ isn't integrable.

For $k = 0, 1, 2, \ldots$ let $A_k = [2k\pi, (2k + 1)\pi]$ and $B_k = [2k\pi + \pi/6, (2k + 1)\pi - \pi/6]$ and define

$$A = \bigcup_{k=0}^{\infty} A_k \ \text{ and } B = \bigcup_{k=0}^{\infty} B_k\,.$$

Then $f^+ = f\chi_A \ge f\chi_B$ and since $\mathrm{Sin}(x) \ge ½$ and $1/x \ge 1/[(2k + 1)\pi - \pi/6]$ on B_k,

$$\frac{Sin(x)}{x} \ge \frac{3}{12k\pi + 5\pi}.$$

Therefore by the monotone convergence theorem and the monotonicity of the Lebesgue integral, we have

$$\int_0^{\infty} f^+ = \int f\chi_A \ge \int f\chi_B \ge \int \sum_{k=0}^{\infty} \frac{3}{12k\pi+5\pi}\chi_{B_k}$$
$$= \sum_{k=0}^{\infty} \frac{3m(B_k)}{12k\pi+5\pi} = \sum_{k=0}^{\infty} \frac{2}{12k+5} = \infty\,.$$

10. By elementary calculus $\lim_{n\to\infty}(1-(x/n))^n = e^{-x}$ for every fixed x. Thus the sequence of functions $f_n(x) = (1-(x/n))^n \chi_{(0,\ n)}(x)$ converges pointwise to e^{-x} on $(0, \infty)$. Clearly each f_n is nonnegative. Furtheremore

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{\left(1-\frac{x}{n+1}\right)^{n+1}}{\left(1-\frac{x}{n}\right)^{n}} = \left(1-\frac{x}{n}\right)\frac{\left(1-\frac{x}{n+1}\right)^{n+1}}{\left(1-\frac{x}{n}\right)^{n+1}}$$
$$= \left(1-\frac{x}{n}\right)\left(1+\frac{x}{(n+1)(n-x)}\right)^{n+1}$$
$$> \left(1-\frac{x}{n}\right)\left(1+\frac{(n+1)x}{(n+1)(n-x)}\right) =$$

$$=\left(1-\frac{x}{n}\right)\left(1+\frac{x}{(n-x)}\right)=1,$$

where in the above estimation we have used Bernoulli's inequality which asserts that $(1+a)^n > 1+na$ whenever a > -1. Hence we see that $f_n < f_{n+1}$. Consequently, by the monotone convergence theorem

$$1=\int_0^\infty e^{-x}dx=\int_{(0,\infty)}\lim_{n\to\infty} f_n=\lim_{n\to\infty}\int_{(0,\infty)} f_n=\lim_{n\to\infty}\int_0^n (1-(x/n))^n\,dx.$$

(Remark: it is important to note that $\int_0^n (1-(x/n))^n\,dx$ can be interpreted as either a Riemann or Lebesgue integral, because the two notions agree for all Riemann integrable functions on bounded intervals. The improper Riemann integral $\int_0^\infty e^{-x}dx$ also agrees with its Lebesgue counterpart, because e^{-x} is bounded and nonnegative; It can be shown via the monotone convergence theorem that the two notions agree whenever this is the case.)

11. As in the previous problem, it is easy to see that the sequence of functions $f_n(x)=(1-(x/n))^n e^{x/2}$ is nonnegative and increasing and so is the sequence $f_n\chi_{(0,\ n)}$. By elementary calculus, $\lim_{n\to\infty} f_n(x)\chi_{(0,\ n)}(x)=e^{-x}e^{x/2}=e^{-x/2}$. Therefore the application of the monotone convergence theorem yields

$$\lim_{n\to\infty}\int_0^n (1-(x/n))^n e^{x/2}dx=\lim_{n\to\infty}\int_{(0,n)} f_n=\int_0^\infty e^{-x/2}dx=2,$$

where the Riemann and Lebesgue integrals in the above calculation agree as explained in the remark at the end of the previous problem.

12. By writing $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, we can reduce the problem to the special case when $f \geq 0$.
The integral

$$\int_0^b g(x)dx=\int_0^b\int_x^b \frac{f(y)}{y}dydx$$

is an iteration of the $\mathbf{R}^2$ integral

$$\int_{R^2} k(x,y)dydx$$

where $k(x,y)=\frac{f(y)}{y}\chi_E(x,y)$ and the set E is the triangle in $\mathbf{R}^2$ defined by

E = $\{(x, y) \in \mathbf{R}^2 : 0 < x < b,\ x \leq y < b\} = \{(x, y) \in \mathbf{R}^2 : 0 < y < b,\ 0 < x < y\}$

By Tonelli's theorem, we can write

$$\int_{R^2} k(x,y)dydx = \int_R \int_R k(x,y)dxdy = \int_0^b \int_0^y \frac{f(y)}{y}dxdy.$$

Since $\int_0^y \frac{f(y)}{y}dx$ is Riemann integrable, the value of the Lebesgue integral agrees with the Riemann integral and equals *f*(y).
Thus,

$$\int_{R^2} k(x,y)dydx = \int_0^b f(y)dy$$

which shows that k(x, y) is Lebesgue integrable. By Fubini's theorem, it then follows that

$$\int_{R^2} k(x,y)dydx = \int_0^b g(x)dx$$

In particular, *g* is Lebesgue integrable and

$$\int_0^b f(y)dy = \int_0^b g(x)dx$$

13. (a) The sequence of functions $g_N(x) = \sum_{n=1}^{N} 2^{-n} f(x - r_n)$ is nonnegative and increasing to *g*. Denoting by f_n the function $f_n(x) = f(x - r_n)$ we therefore obtain via the monotone convergence theorem

$$\int_R g = \int_R \lim_{N\to\infty} g_N = \lim_{N\to\infty} \int_R g_N = \sum_{n=1}^{\infty} \int_R 2^{-n} f_n = \sum_{n=1}^{\infty} 2^{-n} \int_R f$$

where we used the invariance under translation of Lebesgue integrals to conclude that $\int_R f(x + r_n) = \int_R f(x)$. Clearly the Lebesgue integral $\int_R f$ agrees with the improper Riemann integral $\int_0^1 x^{-1/2}dx = 2$ and therefore

$$\int_R g = 2\sum_{n=1}^{\infty} 2^{-n} = 2$$

In particular, *g* (x) < ∞ a.e. x.

(b) *g* is finite a.e. and therefore to show that *g* is discontinuous at every point and unbounded in every interval, it suffices to prove that the oscillation at every point (relative to the subset of **R** on which *g* is finite) equals infinity. More precisely, let

$E_\infty = \{x \in \mathbf{R} : g(x) = \infty\}$. Then $m(E_\infty) = 0$ and we may pick a point $a \in F = \mathbf{R} - E_\infty$. Let $I_\delta = (a-\delta, a+\delta)$ where $\delta > 0$ is small. Define $\omega_g(a : I_\delta) = \sup |g(x) - g(y)|$ where the supremum is taken over all $x, y \in F \cap I_\delta$. Notice that we may pick some rational number r_n in I_δ and since

$$\lim_{x \to r_n} f(x - r_n) = \infty$$

it follows that for each integer m, there is an interval $(r_n, x_m) \subset I_\delta$ such that $f(x-r_n) > m\ 2^n$ for all $x \in (r_n, x_m) \cap F$. Clearly then, $g(x) \geq 2^{-n} f(x - r_n) > m$. Hence $\sup\{|g(x) - g(y)| : x, y \in F \cap I_\delta\} \geq \sup\{|g(x) - g(a)| : x \in F \cap I_\delta\} > m$ and $\omega_g(a : I_\delta) = \infty$. Thus the oscillation of g at a relative to F, $\omega_g(a) = \lim_{\delta\to 0} \omega_g(a : I_\delta) = \infty$ as desired.

It is interesting to note that the removal of a larger set $E \supset E_\infty$ of measure 0 will note tame the oscillation at the remaining points, because $\mathbf{R} - E$ will remain a dense subset of $\mathbf{R}$. Thus, we would still be able to choose $x \in (r_n, x_m) \cap \mathbf{R} - E$ satisfying $g(x) \geq 2^{-n} f(x - r_n) > m$.

(c) Clearly g^2 is finite whenever g is finite. Hence $g^2(x) < \infty$ a.e. x. Since the terms in the series of g are nonnegative, the following inequality may be used to estimate the integral of g^2:

$$g^2(x) = g(x)g(x) = \left(\sum_{n=1}^{\infty} f(x-r_n)\right)\left(\sum_{m=1}^{\infty} f(x-r_m)\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(x-r_n) f(x-r_m) \geq \sum_{n=1}^{\infty} f^2(x-r_n)$$

In any interval (a, b), there is a rational number r_n satisfying $(r_n, c) \subset (a, b)$, where $c = \min\{b, r_n + 1\}$. Consequently

$$\int_{(a,b)} g^2 \geq \int_{(r_n,c)} g^2 \geq \int_{(r_n,c)} \sum_{k=1}^{\infty} f^2(x-r_k) \geq \int_{(r_n,c)} f^2(x-r_n) = \int_0^{c-r_n} f^2(x)dx$$

And since

$$\int_0^{c-r_n} f^2(x)dx = \int_0^{c-r_n} x^{-1}dx = \infty$$

g^2 is not integrable on any interval.

14. Since Cos(x) is a continuous function, by Weierstrass's theorem there exists a sequence of polynomials p_n which converges uniformly to Cos(x) on the interval $[0, 2\pi]$. Specifically, for any $\epsilon > 0$, there is some $N \in \mathbf{N}$ such that

$$\|p_n - Cos\| = \sup_{x\in[0,\ 2\pi]} |p_n(x) - Cos(x)| < \epsilon$$

whenever $n \geq N$.

Since

$$\left|\int_E p_n(x)Cos(x)dx - \int_E Cos^2(x)dx\right| \le \int_E |\, p_n(x) - Cos(x)\,||\, Cos(x)\,|\, dx$$

$$\le \int_E \varepsilon \,|\, Cos(x)\,|\, dx \le \int_E \varepsilon = \varepsilon \; m(E) \le \varepsilon \; m([0,\; 2\pi]) = 2\pi\varepsilon$$

it follows that

$$\lim_{n\to\infty}\int_E p_n(x)Cos(x)dx = \int_E Cos^2(x)dx\,.$$

From the hypothesis that $\int_E x^n Cos(x)dx = 0$ and the additivity of the integral we are lead to conclude $\int_E Cos^2(x)dx = 0$. Now if m(E) > 0, we may pick a closed subset $F \subset E - \{\pi/2, 3\pi/2\}$ with the property m(F) > ½ m(E). Then F is compact as it is closed and bounded and $Cos^2(x)$ attains a minimum value $\delta > 0$ on F. By monotonicity of integration we then have

$$0 = \int_E Cos^2(x)dx \ge \int_F Cos^2(x)dx \ge \int_F \delta = \delta \; m(F) > \frac{\delta}{2} m(E) > 0$$

which is an obvious contradiction. Thus m(E) = 0 as desired.

15. Since $\int_a^b x^{2n} f(x)dx = \int_a^b x^{2n} g(x)dx$ if and only if $\int_a^b x^{2n}(f(x) - g(x))dx = 0$, it suffices to investigate whether $\int_a^b x^{2n} f(x)dx = 0$ (for n = 1, 2, …) implies that f(x) = 0 for all x ∈ [a, b].

The Case [a, b] = [0, 1]: Suppose $\int_0^1 x^{2n} f(x)dx = 0$ for n = 1, 2, … Extend f to a continuous even function $h : [-1, 1] \to \mathbf{R}$ by defining

$$h(x) = \begin{cases} f(x) & x \in [0,\; 1] \\ f(-x) & x \in [-1,\; 0] \end{cases}$$

Then

$$\int_{-1}^1 x^{2n} h(x)dx = \int_{-1}^0 x^{2n} f(-x)dx + \int_0^1 x^{2n} f(x)dx = 2\int_0^1 x^{2n} f(x)dx = 0$$

and

$$\int_{-1}^1 x^{2n+1} h(x)dx = -\int_{-1}^0 x^{2n+1} f(-x)dx + \int_0^1 x^{2n+1} f(x)dx = \int_1^0 x^{2n+1} f(x)dx + \int_0^1 x^{2n+1} f(x)dx = 0\,.$$

In particular, h has both its even and odd moments equal zero. By Weierstrass's theorem, there is a sequence of polynomials p_n which converges uniformly to h on [-1, 1]. Thus

$$0 = \lim_{n\to\infty}\int_{-1}^1 p_n(x)h(x)dx = \int_{-1}^1 h^2(x)dx\,.$$

And since h^2 is nonnegative, h^2 = 0 a.e. and hence h = 0 a.e. However h is continuous and takes the value 0 on a dense subset of [-1, 1], which leads us to conclude that h is identically zero on [-1, 1]. This clearly implies that f (x) = 0 for any x $\in$ [0, 1]. Thus, the even moments of a continuous function on [0, 1] determine the function.

The Case [a, b] = [-1, 1]: Let $f(x) = x$. Then f is not identically 0, but

$$\int_{-1}^{1} x^{2n} f(x)dx = \int_{-1}^{1} x^{2n+1} dx = 0.$$

Thus distinct continuous functions may agree on their even moments.

16. The sequence of Lebesgue integrable functions $\{f_n\}$ increases pointwise to an integrable function f. Therefore, the sequence $\{f_n - f_1\}$ of nonnegative integrable functions increases pointwise to the nonnegative integrable function $f - f_1$. By the monotone convergence theorem we then have

$$\int (f_n - f_1) \to \int (f - f_1).$$

Thus,

$$\int f_n = \int (f_n - f_1) + \int f_1 \to \int (f - f_1) + \int f_1 = \int f.$$

17. Observe that the function $f(x) = \sum_{n=1}^{\infty} a_n \|nx\|^{-\alpha}$ is periodic with period 1. That is,

$f(x+m) = \sum_{n=1}^{\infty} a_n \|n(x+m)\|^{-\alpha} = \sum_{n=1}^{\infty} a_n \|nx+nm\|^{-\alpha} = \sum_{n=1}^{\infty} a_n \|nx\|^{-\alpha} = f(x)$, because the number nx is closest to the integer k if and only if the number nx + nm is closest to the integer k + nm. For n $\in$ **Z**, let E_n = {x $\in$ [n – 1, n]: f (x) = ∞} and set E = {x $\in$ **R**: f (x) = ∞}. If we can prove that m(E_1) = 0, it will follow by periodicity that m(E_n) = 0 for all n and hence that

$m(E) \le \sum_{n=-\infty}^{\infty} m(E_n) = 0$.

By the monotone convergence theorem

$$\int_0^1 f = \sum_{n=1}^{\infty} a_n \int_0^1 \|nx\|^{-\alpha} dx.$$

Let g : [0, 1] → **R** be the function $g(x) = \|x\|$. Then

$$g(x) = \begin{cases} x & x \in [0,\ 1/2] \\ -x+1 & x \in [1/2,\ 1] \end{cases}$$

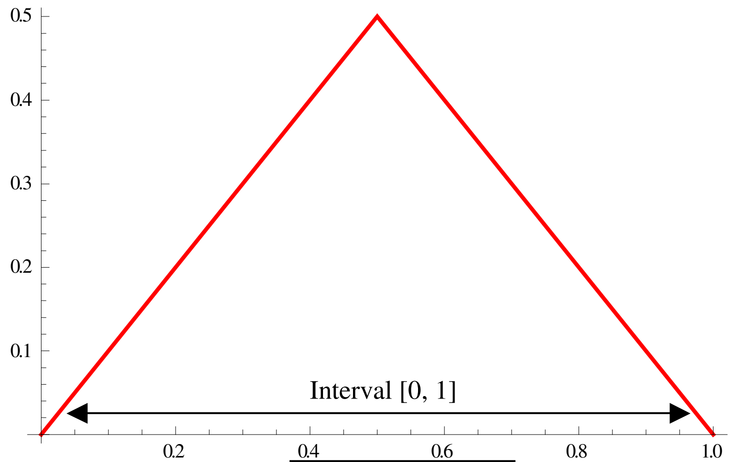
and

$$\int_0^1 \|x\|^{-\alpha} dx = \int_0^1 g^{-\alpha}(x)dx = \int_0^{1/2} x^{-\alpha} dx + \int_{1/2}^1 (-x+1)^{-\alpha} dx.$$

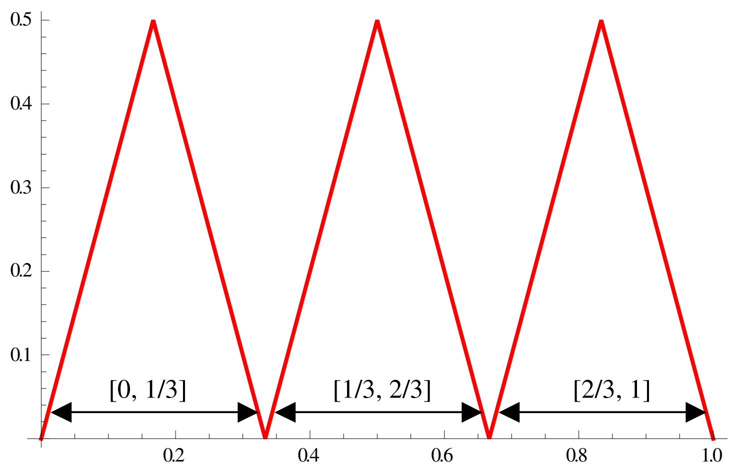
Since 0 < α < 1, the improper Riemann integrals $\int_0^{1/2} x^{-\alpha} dx$ and $\int_{1/2}^1 (-x+1)^{-\alpha} dx$ exist and both equal to $\frac{2^{\alpha-1}}{1-\alpha}$. Hence $\int_0^1 g^{-\alpha}(x)dx = \frac{2^{\alpha}}{1-\alpha}$ and this value agrees with the Lebesgue integral $\int_0^1 g^{-\alpha}$, because all improper nonnegative Riemann integrable functions are also Lebesgue integrable and of the same value.

We use the periodicity of g to evaluate $\int_0^1 \|x\|^{-\alpha}\,dx = \int_0^1 g^{-\alpha}(nx)dx$ as follows:

Note that the graph of g(nx) is a repetition of the graph of g(x) over each interval of the form [(k – 1)/n, k/n]. For convenience, the graphs of g(x) and g(3x) below illustrate the general situation.



Graph of g(x)



Graph of g(3x)

It follows that

$$\int_0^1 g^{-\alpha}(nx)dx = n\int_0^{1/n} g^{-\alpha}(nx)dx = \int_0^1 g^{-\alpha}(x)dx = \frac{2^{\alpha}}{1-\alpha}.$$

Hence

$$\int_0^1 f = \sum_{n=1}^{\infty} a_n \int_0^1 \|nx\|^{-\alpha} dx = \frac{2^{\alpha}}{1-\alpha} \sum_{n=1}^{\infty} a_n < \infty .$$

And since the integral of the nonnegative function f exists, it follows that $m(E_1) = 0$ as desired.

18. The assumption $f \in L^1[0, 1]$ implies f is finite a.e. and therefore for almost every x $\in$ [0, 1], $\lim_{n\to\infty} x^n f(x) = 0$. Since $|x^n f(x)| \le |f| \in L^1[0, 1]$, we may apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = \int_0^1 \lim_{n\to 0} x^n f(x)dx = 0 .$$

19. To calculate $\sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{Sin(x)})^n Cos(x)dx$, it would be convenient to interchange the integral with the sum. That is to say, if we can justify the equality

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{Sin(x)})^n Cos(x)dx = \int_0^{\pi/2} \sum_{n=0}^{\infty} (1-\sqrt{Sin(x)})^n Cos(x)dx ,$$

the problem will reduce to integrating a geometric series that converges for all $x \in (0, \pi/2)$ to the function $\frac{Cos(x)}{\sqrt{Sin(x)}}$ and hence

$$\int_0^{\pi/2} \sum_{n=0}^{\infty} (1-\sqrt{Sin(x)})^n Cos(x)dx = \int_0^{\pi/2} \frac{Cos(x)}{\sqrt{Sin(x)}} dx = \int_0^1 x^{-1/2} dx = 2 ,$$

where we use the agreement of the Lebesgue and improper Riemann integrals for nonnegative Riemann integrable functions.

Finally, to justify interchanging the sum with the integral, define $f_N, f : (0, \pi/2) \to \mathbf{R}$ by $f_N(x) = \sum_{n=0}^{N} (1-\sqrt{Sin(x)})^n Cos(x)$ and $f(x) = \sum_{n=0}^{\infty} (1-\sqrt{Sin(x)})^n Cos(x)$. Clearly the f_N are nonnegative and $f_N \le f_{N+1} \to f$. Therefore, by the monotone convergence theorem, we have

$$\int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} \lim_{N\to\infty} f_N(x)dx = \lim_{N\to\infty} \int_0^{\pi/2} f_N(x)dx = \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^{\pi/2} (1-\sqrt{Sin(x)})^n Cos(x)dx$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{Sin(x)})^n Cos(x)dx .$$

And the desired result is proved.

20. Define for each n and N the set E_N^n = {x $\in \mathbf{R}^d$: |x| $\le$ N, $g_n(x)$ $\le$ N} and set E_N = {x $\in \mathbf{R}^d$: |x| $\le$ N, $g_n(x)$ $\le$ N for all n}. That is $E_N = \bigcap_{n=1}^{\infty} E_N^n$. Several observations are in place.

Observation 1: $g_n \to g$ a.e., where the g_n are nonnegative and therefore 0 ≤ g(x) for almost every x. Furthermore, g(x) ≤ N whenever x ∈ E_N.

Observation 2: The sets E_N are increasing ($E_N \subset E_{N+1}$) and since $g_n \to g$ a.e., the sequence $\{g_n(x)\}$ is Cauchy and therefore bounded for almost every x. Thus the E_N must increase to a measurable subset E ⊂ $\mathbf{R}^d$ where m($\mathbf{R}^d$ – E) = 0.

Observation 3: By hypothesis, $|f_n(x)| \le g_n(x)$ a.e. and $f_n \to f$ a.e., and therefore for almost every x, $g(x) = \lim_{n\to\infty} g_n(x) \ge \lim_{n\to\infty} |f_n(x)| = |f(x)|$. In particular, since g is integrable, so must be f and for all x ∈ E_N, $|f_n(x)| \le$ N and $|f(x)| \le$ N.

Observation 4: For each N

$$\lim_{n\to\infty} \int_{E_N} (g_n - g) = 0$$

This follows from the bounded convergence theorem, since $g_n \to g$ a.e., E_N is a bounded set, and $|g_n(x) - g(x)| \le$ 2N for all x ∈ E_N. Furthermore, using observation 3, we may also conclude that

$$\lim_{n\to\infty} \int_{E_N} |f_n - f| = 0$$

Observation 5: By observation 4 and the hypothesis $\int g_n \to \int g$, it follows that

$$\lim_{n\to\infty} \int_{E_N^c} (g_n - g) = \lim_{n\to\infty} \int (g_n - g) - \lim_{n\to\infty} \int_{E_N} (g_n - g) = 0$$

We are now ready to prove the main result by estimating $\left|\int f_n - \int f\right| = \left|\int (f_n - f)\right|$.

$$\begin{aligned}\left|\int (f_n - f)\right| &\le \int |f_n - f| = \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f| \\ &\le \int_{E_N} |f_n - f| + \int_{E_N^c} g_n + \int_{E_N^c} g \\ &= \int_{E_N} |f_n - f| + \int_{E_N^c} (g_n - g) + 2\int_{E_N^c} g \\ &\le \int_{E_N} |f_n - f| + \left|\int_{E_N^c} (g_n - g)\right| + 2\int_{E_N^c} g\end{aligned}$$

Since g is integrable on $\mathbf{R}^d$, the integral of g decays to zero outside a large bounded set. More precisely, for $\varepsilon > 0$ we may pick N large enough so that

$$\int_{E_N^c} g < \frac{\varepsilon}{4}$$

Holding this N fixed, we note from observations 4 and 5 that for large n

$$\int_{E_N} | f_n - f | \le \frac{\varepsilon}{4}$$

and

$$\left| \int_{E_N^c} g_n - g \right| \le \frac{\varepsilon}{4}$$

Hence $\left|\int (f_n - f)\right| \le \varepsilon$, from which the assertion $\int f_n \to \int f$ readily follows.

21. (a) An easy application of exercise 20 shows the statement to be true:
Set

$$f_n = g_n \chi_{A_n}$$

and rename f = g. Observe that $f_n \to f$ and $| f_n(x) | \le g_n(x)$ a.e. By hypothesis, $g_n \to g$ a.e. and $\int g_n \to \int g$, so all the necessary conditions are satisfied for the modified Lebesgue dominated convergence theorem to apply.

(b) The assumption that g_n is nonnegative a.e. is vital and cannot be dropped. To see this define g_n: $\mathbf{R} \to \mathbf{R}$ and g: $\mathbf{R} \to \mathbf{R}$ by

$$g_n = \frac{1}{n}\chi_{[0,\ e^n]} - \frac{1}{n}\chi_{[-e^n,\ 0)}$$

and

$$g = 0$$

respectively. Also define A_n = [-n, e^n] for n = 1, 2, …

Clearly $g_n \to g$ and this convergence is even uniform. Since $\int g_n = \frac{e^n}{n} - \frac{e^n}{n} = 0$, we have the condition $\int g_n \to \int g$ satisfied as well. Notice however that

$$\int_{A_n} g_n = \int_0^{e^n} \frac{1}{n} - \int_{-n}^{0} \frac{1}{n} = \frac{e^n}{n} - 1 \xrightarrow{as\ n \to \infty} \infty .$$

22. Define the functions $l_n = \inf_{k \ge n} f_k$, $u_n = \sup_{k \ge n} f_k$ and set $l = \liminf_{n\to\infty} f_n$ and $u = \limsup_{n\to\infty} f_n$.
Evidently,

(i) $l_n \le l_{n+1}$ and $\lim_{n\to\infty} l_n = l$,

(ii) $u_n \ge u_{n+1}$ and $\lim_{n\to\infty} u_n = u$,

,and since $|f_k| \le g$ for all k,

(iii) $|l_n| \le g$ and $|u_n| \le g$ a.e. for all n.

Since g is integrable, the application of Lebesgue dominated convergence theorem leads to the conclusions

$$\int (\liminf_{n\to\infty} f_n) = \int \lim_{n\to\infty} l_n = \lim_{n\to\infty} \int l_n \qquad (1)$$

and

$$\int (\limsup_{n\to\infty} f_n) = \int \lim_{n\to\infty} u_n = \lim_{n\to\infty} \int u_n \qquad (2).$$

Monotonicity of the integral and the fact that $l_n \le f_n \le u_n$ for every n imply

$$\lim_{n\to\infty} \int l_n = \liminf_{n\to\infty} \int l_n \le \liminf_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int u_n = \lim_{n\to\infty} \int u_n,$$

where we use the fact that $\int l_n$ and $\int u_n$ converge in deriving $\lim_{n\to\infty} \int l_n = \liminf_{n\to\infty} \int l_n$ and $\limsup_{n\to\infty} \int u_n = \lim_{n\to\infty} \int u_n$ respectively. Putting this inequality with identities (1) and (2), we get the statement of the exercise.

23. (a) Suppose that $\int_E f = 0$ for all measurable E ⊂ [0, 1] with m(E) = ½. Set A = {x ∈ [0, 1]: f(x) ≥ 0} and B = {x ∈ [0, 1]: -f(x) ≥ 0}. Then [0, 1] = A ∪ B and one of the sets A or B must have outer measure greater than or equal to ½. We show that m(A) ≥ ½ implies f = 0 on [0, 1] a.e. This will establish the desired result, since if m(A) < ½, we could switch to –f.
Assume m(A) ≥ ½. Since A is measurable, by exercise 20 in the measure theory problem list we can pick a, b ∈ (0, 1) such that m([0, a] ∩ A) = ½ and m([b, 1] ∩ A) = ½. Then a ≥ b, for otherwise

1 = m([0, 1]) > m([0, a]) + m([b, 1]) ≥ m([0, a] ∩ A) + m([b, 1] ∩ A) = 1,

which is a contradiction. Hence, in particular, A = ([0, a] ∩ A) ∪ ([b, 1] ∩ A). For convenience, we label E = [0, a] ∩ A and F = [b, 1] ∩ A in the calculations that follow.
Since f is nonnegative on A, an easy application of the hypothesis to the sets E and F yields

$$\int_A f \le \int_E f + \int_F f = 0.$$

Consequently, f(x) = 0 for almost every x ∈ A. And since [0, 1] = [0, ½] ∪ [½ , 1], we have

$$\int_B f = \int_0^1 f - \int_A f = \int_0^{1/2} f + \int_{1/2}^1 f - \int_A f = 0 + 0 - 0 = 0 .$$

As f is nonpositive on B, it follows that f (x) = 0 for almost every x ∈ B. The observation {x ∈ [0, 1]: f (x) = 0} = {x ∈ A : f (x) = 0} ∪ {x ∈ B : f (x) = 0} leads to the conclusion that m{x ∈ [0, 1]: f (x) = 0} = 0 and therefore to the desired result.

(b) Now assume that f > 0 a.e. and define K = {x ∈ [0, 1]: f (x) = 0}, K_n = {x ∈ [0, 1]: 0 ≤ f (x) < 1/n}, and H_n = {x ∈ [0, 1]: f (x) ≥ 1/n}. Then for all n, K_n and H_n are disjoint and [0, 1] = $K_n \cup H_n$. Furthermore, the sets K_n are finite in measure and decreasing ($K_n \supset K_{n+1}$) to the limit K with m(K) = 0. In particular, $\lim_{n\to\infty}$ m(K_n) = m(K) = 0 and it follows that m(K_n) < ¼ for all large enough n. Fix one such n and define

$$\varphi(x) = 0 \cdot \chi_{K_n}(x) + \frac{1}{n}\chi_{H_n}(x)$$

for all x ∈ [0, 1]. Then φ is nonnegative and $f(x) \ge \varphi(x)$ for every x in [0, 1]. If E ⊂ [0, 1] with m(E) ≥ ½,

$$\int_E f \ge \int_E \varphi = \int_{E\cap H_n} \varphi = \frac{1}{n} m(E \cap H_n) .$$

Notice that m(E ∩ H_n) = m(E) – m(E ∩ K_n) ≥ ¼ . Thus, inf{$\int_E f$: m(E) ≥ ½} ≥ 1/(4n) > 0.

24. For parts (a) through (c) it is helpful to note that $0 \le (1-nx)^2 = 1 + n^2x^2 - 2nx$ and therefore $2nx \le 1 + n^2x^2$. For part (d) we will use the observation $\left|\frac{u^3}{1+u^4}\right| \le \frac{3^{3/4}}{4}$ for all u ∈ **R**, which can be verified with elementary calculus.

(a) For all x ∈ (0, 1], $\frac{nx}{1+n^2x^2} \le \frac{nx}{2nx} = 1/2$. Thus the $f_n(x) = \frac{nx}{1+n^2x^2}$ are uniformly bounded on the compact support [0, 1] by M = ½. And since $f_n \to 0$ pointwise, it follows from the Lebesgue bounded convergence theorem that $\int_0^1 f_n \to 0$.

Alternatively, observe that

$$\int_0^1 f_n = \frac{1}{2n}\int_0^1 \frac{2n^2x}{1+n^2x^2}dx$$

which, after simple u-substitution, reduces to

$$\frac{1}{2n}\int_0^{n^2}\frac{1}{1+u}\,dx=\frac{\ln(1+n^2)}{2n}\to 0\,.$$

(b) For all x ∈ (0, 1], $f_n(x)=\dfrac{n\sqrt{x}}{1+n^2x^2}=\dfrac{nx}{1+n^2x^2}\dfrac{1}{\sqrt{x}}\le\dfrac{1}{2}x^{-1/2}$ by the same estimation done in part (a). Since $\frac{1}{2}x^{-1/2}$ is Lebesgue integrable on [0, 1] and since $\lim_{n\to\infty}\dfrac{n\sqrt{x}}{1+n^2x^2}=0$ for every x, it follows from the Lebesgue dominated convergence theorem that $\int_0^1 f_n\to 0$.

(c) By elementary calculus, ln(u) ≤ u for all u ≥ 1. Hence,
$|f_n(x)|=\left|\dfrac{nx\ln(x)}{1+n^2x^2}\right|=\left|\dfrac{-2nx\ln(x^{-1/2})}{1+n^2x^2}\right|\le\dfrac{2n\sqrt{x}}{1+n^2x^2}$. The squeeze theorem together with part (b) implies $\lim_{n\to\infty}\left|\int_0^1 f_n\right|\le\lim_{n\to\infty}\int_0^1|f_n(x)|\,dx\le\lim_{n\to\infty}2\int_0^1\dfrac{n\sqrt{x}}{1+n^2x^2}dx=0$.

(d) Notice that for all x ∈ (0, 1], $\dfrac{n^{3/2}x}{1+n^2x^2}=\dfrac{(nx)^{3/2}}{1+(nx)^{4/2}}\cdot\dfrac{1}{\sqrt{x}}=\dfrac{u^3}{1+u^4}\cdot x^{-1/2}\le\dfrac{3^{3/4}}{4}x^{-1/2}$, where we set u = nx. Since $\lim_{n\to\infty}\dfrac{n^{3/2}x}{1+n^2x^2}=0$, the integral converges to 0 by Lebesgue dominated convergence theorem.
We can also verify the result with direct computation:

$$\int_0^1\frac{n^{3/2}x}{1+n^2x^2}dx=\frac{1}{2\sqrt{n}}\int_0^1\frac{2n^2x}{1+n^2x^2}dx=\frac{1}{2\sqrt{n}}\int_0^{n^2}\frac{1}{1+u}du=\frac{\ln(1+n^2)}{2\sqrt{n}}\to 0$$

25. (a) Let $f_n(x)=\dfrac{Sin(e^x)}{1+nx^2}$ and $g(x)=\dfrac{1}{1+x^2}$. Then it is immediate that for all x ∈ [0, ∞) and all n, $|f_n(x)|$ ≤ g(x). Since $f_n(x)\to 0$ for all x ≠ 0 and since $\int_0^\infty g(x)dx=\dfrac{\pi}{2}<\infty$, we may apply Lebesgue dominated convergence theorem to conclude that $\lim_{n\to\infty}\int_0^\infty\dfrac{Sin(e^x)}{1+nx^2}dx=0$.

(b) This sequence of integrals converges to 0 as well, but this is not as straight forward to demonstrate. Observe that $f_n(x) = \dfrac{nCos(x)}{1+n^2x^{3/2}}$ is nonnegative in the interval [0, 1] with $\lim_{n\to\infty} f_n(x) = 0$ for all x $\in$ (0, 1]. Moreover, for any x < y, $f_n(x) > f_n(y)$ and so each function in the sequence is decreasing. In particular, $f_n \to 0$ uniformly on any interval of the form [*a*, 1] where 0 < *a*. Uniform convergence allows the interchange of integral and limit to conclude that

$$\lim_{n\to\infty}\int_a^1 f_n = \int_a^1 \lim_{n\to\infty} f_n = 0.$$

Denote by g_n: [0, 1] $\to$ **R**, the sequence of functions $g_n(x) = \dfrac{n}{1+n^2x^2}$ and notice that

$$f_n(x) \le \frac{n}{1+n^2x^{3/2}} \le \frac{n}{1+n^2x^{4/2}} = g_n(x).$$

Hence

$$\limsup_{n\to\infty}\int_0^1 f_n(x)dx = \limsup_{n\to\infty}\left(\int_0^a \frac{nCos(x)}{1+n^2x^{3/2}}dx + \int_a^1 \frac{nCos(x)}{1+n^2x^{3/2}}dx\right)$$

$$\le \limsup_{n\to\infty}\int_0^a g_n(x)dx + \limsup_{n\to\infty}\int_a^1 \frac{nCos(x)}{1+n^2x^{3/2}}dx$$

$$= \limsup_{n\to\infty}\int_0^a \frac{n}{1+n^2x^2}dx = \tan^{-1}(a).$$

Now $\lim_{a\to 0}\tan^{-1}(a) = 0$ and we can choose *a* arbitrarily close to 0 to establish that $\limsup_{n\to\infty}\int_0^1 f_n(x)dx$ = 0. Since we are integrating nonnegative functions, the argument is complete.

26. Observe that $f_n(x)$ < 0 if and only if $a^2e^{-nax} < b^2e^{-nbx}$, which happens if and only if x < $\dfrac{2\ln(b/a)}{n(b-a)}$ as can be verified by moving the exponential functions to the one side and the constants to the other side of the inequality. Set $c = \dfrac{2\ln(b/a)}{(b-a)}$ and $c_n = c/n$. Then

$$\int_0^\infty |f_n| = \int_0^{c_n} -f_n + \int_{c_n}^\infty f_n = \frac{(b-a)-2(be^{-bc}-ae^{-ac})}{n} = \frac{p}{n}.$$

And therefore,

$$\sum_{n=1}^\infty\int_0^\infty |f_n| = \sum_{n=1}^\infty \frac{p}{n} = \infty.$$

A similar result is obtained when we calculate $\sum_{n=1}^{\infty}\int_0^{\infty} f_n$:

$$\sum_{n=1}^{\infty}\int_0^{\infty} f_n = \left.\frac{-ae^{-nax}+be^{-nbx}}{n}\right|_0^{\infty} = \sum_{n=1}^{\infty}\frac{b-a}{n} = \infty .$$

However, when we interchange the sum and product, we get

$$\int_0^{\infty}\sum_{n=1}^{\infty} f_n = \int_0^{\infty}\left(\frac{a^2e^{-ax}}{1-e^{-ax}} - \frac{b^2e^{-bx}}{1-e^{-bx}}\right)dx = \ln\left.\left(\frac{(1-e^{-bx})^b}{(1-e^{-ax})^a}\right)\right|_0^{\infty} = 0 .$$

27. (a) Define $f_n : [0, \infty) \to \mathbf{R}$ by $f_n(x) = \dfrac{nSin(x/n)}{x(1+x^2)}$. Then $\lim_{n\to\infty} f_n(x) = \dfrac{1}{1+x^2}$ and since $\left|\dfrac{Sin(x/n)}{(x/n)}\right| \le 1$, it follows that $|f_n(x)| \le \dfrac{1}{1+x^2} = g(x)$. Applying the Lebesgue dominated convergence theorem therefore yields

$$\lim_{n\to\infty}\int_0^{\infty} f_n = \int_0^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

(b) Define $f_n : [0, 1] \to \mathbf{R}$ by $f_n(x) = \dfrac{1+nx^2}{(1+x^2)^n}$ and observe that for n ≥ 2, each function in the sequence is decreasing. Furthermore, $|f_n(x)| \le |f_n(0)| = 1$ for all n ≥ 2. And since $\lim_{n\to\infty} f_n(x) = 0$ for all x > 0, $f_n \to 0$ a.e. We may therefore apply the bounded convergence theorem to conclude

$$\lim_{n\to\infty}\int_0^1 f_n = \lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}dx = \int_0^1 \lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n}dx = 0 .$$

(c) Let $f_n : [0, \infty) \to \mathbf{R}$ be given by $f_n(x) = \dfrac{Sin(x/n)}{(1+x/n)^n}$ and define $g_n(x) = \dfrac{1}{(1+x/n)^n}$. Then if we hold x fixed, $\lim_{n\to\infty} f_n(x) = 0$, $\lim_{n\to\infty} g_n(x) = e^{-x}$, and $|f_n| \le |g_n|$. Moreover, if n > 1,

$$\int_0^{\infty} g_n(x)dx = \int_0^{\infty}(1+x/n)^{-n}dx = -\left.\frac{n(1+x/n)^{-n+1}}{n-1}\right|_0^{\infty} = \frac{n}{n-1},$$

and therefore $\lim_{n\to\infty}\int_0^{\infty} g_n = 1$.

Notice also that $\int_0^\infty e^{-x}dx = 1$. Hence if we label f (x) = 0 and g(x) = e^{-x}, we have $f_n \to f$ a.e., $g_n \to g$ a.e., and $g_n \geq |f_n|$ a.e., which are the necessary conditions in applying the version of Lebesgue dominated convergence theorem that was discussed in exercise 20. In particular, we can now conclude that

$$\lim_{n\to\infty}\int_0^\infty \frac{Sin(x/n)}{(1+x/n)^n}dx = \int_0^\infty \lim_{n\to\infty}\frac{Sin(x/n)}{(1+x/n)^n}dx = 0.$$

(d) Define f_n: [0, ∞) → **R** by $f_n(x) = \dfrac{n}{1+n^2x^2}$ and note that the value of $\lim_{n\to\infty}\int_a^\infty f_n$ clearly depends on a. Despite this, whatever may be the value of a is selected, the integrand is nonnegative and therefore the techniques of Riemann integration from elementary calculus may be used to evaluate this Lebesgue integral, since both integrals yield the same results. With the use of simple u – substitution, we obtain

$$\lim_{n\to\infty}\int_a^\infty f_n = \lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}dx = \lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}dx = \lim_{n\to\infty}\tan^{-1}(nx)\Big|_a^\infty = \lim_{n\to\infty}\left(\frac{\pi}{2} - \tan^{-1}(na)\right).$$

Hence, if $a > 0$, $\lim_{n\to\infty}\int_a^\infty f_n = 0$, if $a = 0$, $\lim_{n\to\infty}\int_a^\infty f_n = \pi/2$, and if $a < 0$, $\lim_{n\to\infty}\int_a^\infty f_n = \pi$.

28. Observe first that for any $a > 0$, the function $p(x) = (1 + a/x)^x$ is increasing for all x > 0 and $\lim_{x\to\infty} p(x) = e^a > 1$. Thus the sequence of functions $f_n(x) = n\ln(1 + f(x)/n) = \ln(1 + f(x)/n)^n$ increases pointwise to $f = \ln(e^f)$. By the monotone convergence theorem, it therefore follows that $\lim_{n\to\infty}\int_E n\ln(1 + f/n) = \int_E f$.
With the added assumption that f is not 0 a.e. on E, the expression $n\ln(1+(f/n)^{1/2}) = \sqrt{n}\,\ln(1+\sqrt{f}/\sqrt{n})^{\sqrt{n}}$ increases to infinity, since it is of the form $\left(\sqrt{f}\right)^\infty$ and $\sqrt{f} > 1$ on a subset K ⊂ E of positive measure. Again, by the monotone convergence theorem, $\lim_{n\to\infty}\int_E n\ln(1+(f/n)^{1/2}) \geq \int_K \infty = \infty$.

29. Before proceeding to the problem at hand, it would be helpful to review the following points:

Observation 1: The output of the Lebesgue integral over a bounded interval [a, b] is the same as the output of the Riemann integral when the input happens to be a Riemann integrable function. In particular, $(L)\int_a^b f = (R)\int_a^b f$, whenever f is continuous on [a, b].

Observation 2: If f is nonnegative, the monotone convergence theorem easily implies that $(L)\int_a^b f = (R)\int_a^b f$ for any choice of a and b in [-∞, ∞].

Observation 3: If the Riemann integral of f over [a, b] is absolutely convergent, $(L)\int_a^b f = (R)\int_a^b f$ for any choice of a and b in [-∞, ∞] as can be justified with the application of the Lebesgue dominated convergence theorem.

We are now ready to deal with the problem. Let $f(x) = x^\alpha Sin(x^\beta)$. We investigate $\int_0^1 f(x)dx$ by breaking the problem into 3 cases.

Case 1: (α > - 1, $\beta \in$ R.) $(R)\int_0^1 |f(x)|dx \le (R)\int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$. Thus, since f is nonnegative and since the Riemann integral of f is absolutely convergent, it follows from observation 2 that $(L)\int_0^1 f = (R)\int_0^1 f = \frac{1}{\alpha+1}$ for all α > - 1.

Case 2: ($\alpha \le$ - 1, $\beta \ge$ 0.) Write $f(x) = x^\alpha Sin(x^\beta) = x^{\alpha+\beta}\frac{Sin(x^\beta)}{x^\beta}$ and fix $\varepsilon > 0$ such that $\left|\frac{Sin(x^\beta)}{x^\beta} - 1\right| < \frac{1}{2}$. Then

$$\frac{1}{2}(R)\int_0^\varepsilon x^{\alpha+\beta}dx + (R)\int_\varepsilon^1 x^\alpha Sin(x^\beta)dx$$

$$\le (R)\int_0^1 |f(x)|dx$$

$$\le \frac{3}{2}(R)\int_0^\varepsilon x^{\alpha+\beta}dx + (R)\int_\varepsilon^1 x^\alpha Sin(x^\beta)dx$$

Thus, by integral comparison test, $(R)\int_0^1 |f(x)|dx$ converges if and only if $(R)\int_0^\varepsilon x^{\alpha+\beta}dx$ converges, which happens if and only if $\alpha + \beta$ > - 1. As f is nonnegative, we know from observation 2 that $(L)\int_0^1 f = (R)\int_0^1 f$ and that these integrals are finite under the condition $\alpha + \beta$ > - 1.

Case 3: ($\alpha \le$ - 1, β < 0.) Define $f_n = f\chi_{[1/n,\ 1]}$. Clearly the f_n are continuous (and hence Riemann integrable and Lebesgue integrable) functions on [0, 1] with $f_n \to f$ pointwise a.e. Consequently we are free to apply techniques of elementary calculus to f_n to investigate the Riemann and Lebesgue integrability of f. Making a u-substitution with $u = x^{\beta}$ to the integral $\int_0^1 f_n = \int_{1/n}^1 x^{\alpha} Sin(x^{\beta})dx$ and taking the limit as n goes to infinity leads to the investigation of the integral $\frac{1}{-\beta}\int_1^{\infty}\frac{Sin(u)}{u^r}du$ with $r = 1 + \frac{\alpha+1}{-\beta}$. From the hypothesis on α and β, we know that $r \le 1$. Adopting the procedure of exercise 9, we can convert $\frac{1}{-\beta}\int_1^{\infty}\frac{Sin(u)}{u^r}du$ into a convergent alternating series under the condition $0 < r \le 1$. Thus $(R)\int_0^1 f$ exists under the assumption $\beta - 1 < \alpha < -1$. On the other hand, mimicking the proof that $\int_0^{\infty}\frac{Sin(x)}{x}dx$ is not Lebesgue integrable, leads to the conclusion that $\frac{1}{-\beta}\int_1^{\infty}\frac{Sin(u)}{u^r}du$ diverges as a Lebesgue integral for all α and β that satisfy the hypothesis of case 3.

30. To investigate whether $f(x) = \sum_{n=1}^{\infty} xn^{-\alpha}e^{-nx}$ is continuous we consider 2 cases.

Case 1: ($\alpha > 0$) Write $f(x) = \sum_{n=1}^{\infty} n^{-\alpha-1}(nx)e^{-nx}$ and apply the Weierstrass M –test, which yields $\sum_{n=1}^{\infty} n^{-\alpha-1}\sup_{x\ge 0}\left|nxe^{-nx}\right| = \sum_{n=1}^{\infty} n^{-\alpha-1}e^{-1}$. Since the latter series is absolutely convergent for $\alpha > 0$, f must be the uniform limit of the sequence $f_N(x) = \sum_{n=1}^{N} xn^{-\alpha}e^{-nx}$ of continuous functions. Therefore f is itself continuous.

Case 1: ($\alpha \le 0$) Observe that $f(0) = 0$ for all choices of $\alpha \in \mathbf{R}$ and that whenever x > 0 and $\alpha \le 0$ we have $f(x) = \sum_{n=1}^{\infty} xn^{-\alpha}e^{-nx} \ge \sum_{n=1}^{\infty} xe^{-nx} = \frac{xe^{-x}}{1-e^{-x}} = g(x)$. By l'Hospital's rule,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{e^{-x}(1-x)}{e^{-x}} = 1.$$

Therefore,

$$\limsup_{x\to 0^+} f(x) \ge \limsup_{x\to 0^+} g(x) = 1,$$

which shows that f cannot be continuous at x = 0 when $\alpha \le 0$.

For any $\alpha \in \mathbf{R}$, we observe that f is the monotonically increasing limit of the sequence of nonnegative functions $f_N(x) = \sum_{n=1}^{N} xn^{-\alpha}e^{-nx}$. Therefore, by the monotone convergence theorem,

$$\int_0^\infty f = \int_0^\infty \left(\sum_{n=1}^\infty xn^{-\alpha} e^{-nx}\right) dx = \sum_{n=1}^\infty \int_0^\infty xn^{-\alpha} e^{-nx} dx = \sum_{n=1}^\infty n^{-\alpha-2} .$$

Thus the integral converges if and only if the series $\sum_{n=1}^\infty n^{-\alpha-2}$ converges, which gives α > - 1 as the condition for membership in $L^1[0, \infty)$.

31. (a) Notice that $f(x) = \sum_{n=1}^\infty (1/n) Exp(-n(x-n)^2)$ is nonnegative on **R** and is the limit of a monotonically increasing sequence of nonnegative functions $f_N(x) = \sum_{n=1}^N (1/n) Exp(-n(x-n)^2)$. Thus the monotone convergence theorem implies that

$$\int_{-\infty}^\infty f = \sum_{n=1}^\infty \frac{1}{n} \int_{-\infty}^\infty e^{-n(x-n)^2} dx = \sum_{n=1}^\infty \frac{\sqrt{\pi}}{n^{3/2}} .$$

The series is convergent and therefore $f \in L^1(\mathbf{R})$.

(b) The function f is continuous on **R** if and only if it is continuous on every interval of the form (-m, m), m $\in$ **N**. Set $k_n(x) = (1/n) Exp(-n(x-n)^2)$ and notice that on (-m, m)

$$\sup_{x \in (-m,\ m)} |k_n| = \begin{cases} 1/n & if \ \ n \le m \\ (1/n) e^{-n(m-n)^2} & if \ \ n > m \end{cases}$$

Applying the Weierstrass M-test we obtain the estimate

$$\sup_{x \in (-m,\ m)} |f| \le \sum_{n=1}^m \frac{1}{n} + \sum_{n=m+1}^\infty \frac{1}{n} e^{-n(m-n)^2} \le \sum_{n=1}^m \frac{1}{n} + \sum_{n=m+1}^\infty e^{-n} < \infty .$$

The M-test shows that f is the uniform limit of a continuous sequence of functions $f_N(x) = \sum_{n=1}^N (1/n) Exp(-n(x-n)^2)$ on (-m, m). Therefore f is continuous on (-m, m) and hence, continuous on **R**.

(c) We proceed to examine differentiability of f on (-m, m) and make the preliminary observation that if the sequence $f_N{}'$ is uniformly convergent, $f_N{}'$ must converge to f' (For a justification of this fact, consult theorem 10.7. on page 152 in the Carothers textbook).

Define $g(x) = \sum_{n=1}^\infty -2(x-n) Exp(-n(x-n)^2) = \sum_{n=1}^\infty k_n{}'(x)$ and notice that on (-m, m)

$$\sup_{x \in (-m,\ m)} |k_n{}'| = \begin{cases} 2(2ne)^{-1/2} & if \ \ n + (2n)^{-1/2} \le m \\ 2(n-m) e^{-n(m-n)^2} & if \ \ n > m \end{cases}$$

Applying the Weierstrass M-test we obtain the estimate

$$\sup_{x \in (-m,\ m)} |\, g\, | \le \sum_{n=1}^{m} 2(2ne)^{-1/2} + \sum_{n=m+1}^{\infty} 2(n-m)e^{-n(m-n)^2} \le \sum_{n=1}^{m} 2(2ne)^{-1/2} + \sum_{n=m+1}^{\infty} ne^{-n} < \infty\,.$$

The M-test shows that g is the uniform limit of a continuous sequence of functions $f_N\,'(x) = \sum_{n=1}^{N} -2(x-n)Exp(-n(x-n)^2)$ on (-m, m). By theorem 10.7, g = f' and therefore f is everywhere differentiable.